INVESTIGATION OF TRANSITION FROM SUBCRITICAL TO SUPERCRITICAL REGIME OF VISCOUS-INVISCID INTERACTION IN THE WAKE BEHIND A FLAT PLATE

A. A. Kovalenko and I. I. Lipatov UDC 532.526

1. In classical Prandtl theory it is postulated that the outer inviscid flow is independent of flow in the boundary layer. However, this postulate is not always valid. For example, in some conditions of hypersonic flow over bodies one observes strong interdependence between the inviscid and the viscous flows (the moderate and strong interaction regimes [1]).

The mathematical formulation for the strong viscous-inviscid interaction regime differs from that of the classical boundary layer problem. The parabolic system of boundary layer equations allowing for the induced pressure gradient interaction has the property of weak ellipticity. This property is the mathematical reflection of the physical process of propagation of perturbations upstream. Study of solutions of flow in the strong interaction regime has shown that perturbations can be propagated upstream to distances comparable with the length of the wetted body. Flows with this type of propagation of perturbations are conventionally called flows with interaction of subcritical type [2]. However, one also observes situations where the extent of the propagation of perturbations is limited to several boundary layer thicknesses. This localization of perturbations is typical of flows with interaction of supercritical type [3]. Investigations have shown that the parameter governing the nature of propagation of perturbations, and thereby the type of interaction, is a certain mean integral Mach number in the boundary layer. This parameter, obtained earlier in the study of internal inviscid flows, is called the Pearson number [4].

In conditions where the viscous-inviscid interaction plays a governing role the flow can have interaction regions of both subcritical and supercritical type. This situation is observed in symmetric hypersonic flow over a flat plate of finite length and zero thickness, investigated in this study. Here subcritical interaction is found in the boundary layer over the plate and at some distance in the wake behind it. The gradual acceleration of the gas under the influence of viscous forces near the trailing edge leads to conditions where at some section of the wake the mean integral Mach number becomes equal to 1 (the Pearson number goes to zero). Downstream of this section one finds the supercritical type of interaction.

This process has much in common with one-dimensional flow of an inviscid gas in a Laval nozzle, where the flow is accelerated smoothly to be supersonic at the nozzle exit. From the theory of one-dimensional flow of an inviscid gas it is known also that an excess pressure over the nominal value at the nozzle exit leads to transition of the flow from supersonic to subsonic, with a shock wave forming downstream of the nozzle exit. Further increase of pressure is accompanied by displacement of the shock up to the nozzle throat.

In this paper we have studied how increasing pressure in the wake section corresponding to the right hand boundary of the computed region affects the nature of the entire flow. A pressure increase in the wake can be caused by the presence of a body in the wake behind a flat plate or by a compression shock incident on the wake. It has been determined that by assigning a wake pressure exceeding a certain value one can formulate a solution that can be treated as discontinuous. This discontinuity is accompanied by a change of the type of interaction to subcritical with a subsequent smooth pressure transition to the assigned value. Thus, in this aspect one observes the analogy between hypersonic flow in the wake and flow in the nozzle [5]. One should remember that the discontinuity obtained is not the usual gasdynamic shock. In spite of the analogy with one-dimensional inviscid flow, the flow in the wake is substantially two-dimensional. And only the mean characteristics of this flow, averaged over the wake cross sections, justify it being considered as subcritical (subsonic) or supercritical (supersonic). Taking account of the above statements, when the term discontinuity is used in this paper it should be considered as being in quotation marks.

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For the weak interaction regime a discontinuous solution was obtained in [6]. It was shown that for small values of the surface temperature factor the interaction type cannot change due to the small pressure drop. This is due to the nonlinear variation of the functions in the wall sublayer against the background of the linear variation in the main part of the boundary layer. In contrast with [6] in this paper we have studied regimes where a finite pressure drop causes a nonlinear variation over the entire boundary layer thickness.

2. We now consider the problem of hypersonic flow over a flat plate of zero thickness and in the symmetrical wake behind it. We postulate that the characteristic Reynolds number Re has a large but subcritical value, so that the flow in the boundary layer and in the wake is laminar. We postulate that the regime is the strong viscous–inviscid interaction, which is valid for high enough values of incident flow Mach number.

Everywhere below the superscript 0 denotes dimensional values, and subscripts ∞ and c denote parameters of the incident stream and characteristic values.

To derive the dimensionless equations of the hypersonic boundary layer one must normalize all the variables appearing in the original full Navier–Stokes equations by the appropriate characteristic values, according to the rule $\Phi^0(x^0, y^0) = \Phi_c{}^0\Phi(x, y)$. By normalization in this case we mean the derivation of dimensionless variables, allowing for their order of magnitude in the flow region considered. The normalizing factors $\Phi_c{}^0$ depend on the plate length ℓ^0, the velocity $u_\infty{}^0$, and the density $\rho_\infty{}^0$ of the incident stream, and the small parameter $d_c \ll 1$, which is the characteristic slope angle of the outer edge of the unperturbed boundary layer. According to [1] the following values are characteristic: for the rectangular coordinates – $x_c{}^0 = \ell^0$, $y_c{}^0 = d_c\ell^0$, for the velocity components – $u_c{}^0 = u_\infty{}^0$, $v_c{}^0 = d_c u_\infty{}^0$, for the density – $\rho_c{}^0 = d_c{}^2\rho_\infty{}^0$, for the static pressure – $p_c{}^0 = d_c{}^2\rho_\infty{}^0 u_\infty{}^{02}$, and for the total enthalpy – $h_c{}^0 = u_\infty{}^{02}/2$. It can be shown that in this normalization for the total enthalpy it is appropriate to use the quantity $t_c{}^0 = u_\infty{}^{02}/2c_p{}^0$ ($c_p{}^0$ is the specific heat at constant pressure) as the characteristic value of absolute temperature. Then, taking into account the power law dependence of dynamic viscosity on absolute temperature adopted here we need to know the quantity $\mu_c{}^0 = \mu^0(t_c{}^0)$ as a characteristic for it.

The hypersonic boundary layer equations can be obtained by substituting these normalized variables into the full Navier–Stokes equations. To do this it is enough to accomplish the limiting transition for $Re = \rho_\infty{}^0 u_\infty{}^0 \ell^0/\mu_c{}^0 \to \infty$, requiring here that the convective term and the main viscous term of the momentum equation have the same order of magnitude. The typical slope angle of the outer edge of the boundary layer, for which this condition holds, is linked to the Reynolds number by the relation $d_c = Re^{-1/4}$.

The system of equations and the boundary conditions for the viscous part of the shock layer in normalized variables takes the form (the subscripts x and y denote partial derivatives with respect to the corresponding variable):

$$
\begin{gathered}
(\rho u)_x + (\rho v)_y = 0,\ \rho u u_x + \rho v u_y + p_x = (\mu u_y)_y, \\
\rho u h_x + \rho v h_y = \frac{1}{\Pr}(\mu h_y)_y + \left(1 - \frac{1}{\Pr}\right)[\mu (u^2)_y]_y, \\
p = \frac{\varkappa - 1}{2\varkappa}\rho (h - u^2),\ \mu = (h - u^2)^\omega,\ d = \int_0^{y_L} dy, \\
\begin{array}{lll}
y = 0: & v = u = h - h_B = 0, & 0 \leqslant x \leqslant 1, \\
& v = u_y = h_y = 0, & 1 < x, \\
y = y_L: & u = h = 1, & 0 \leqslant x.
\end{array}
\end{gathered}
\tag{2.1}
$$

Here κ is the specific heat ratio for a perfect gas; Pr is the Prandtl number; ω is the exponent in the dependence of dynamic viscosity on temperature (later on $\omega = 1$); the subscripts B and L refer to values of the variables on the plate surface and at the outer edge of the viscous layer.

A relation to close the system (2.1) can be obtained using the tangent wedge method [1] widely applied in practical computations. This method links the pressure with the local slope of the stream lines from the incident flow direction. According to the conventional concept of an "efficient" body, this stream line slope is formed by the pulling action of the plate boundary layer and the wake behind the plate. In the strong viscous–inviscid interaction regime postulated here is it easy to obtain a relation for the pressure:

$$p = \frac{\varkappa+1}{2}(d_x)^2. \tag{2.2}$$

To solve the problem on a computer it is appropriate to introduce dimensionless variables which will not only eliminate the boundary layer density from the equations but will also assign a rectangular shape to the computing region. From the computing viewpoint it is also convenient to use dependent variables accounting for the nature of the behavior of functions in the immediate vicinity of the plate leading edge and the well-known Lees–Stewartson similarity solution [1]. These requirements are satisfied by the variables

$$X = x, \quad Y = c_1 x^{-1/4}\int_0^y \rho dy, \quad U = u, \quad H = h, \quad P = x^{1/2}p,$$
$$V = x(uY_x + c_1 x^{-1/4}\rho v),\ D = x^{-3/4}d,\ D_L = x^{-3/4}d_L.$$

Written in these variables the system (2.1), Eq. (2.2), the boundary conditions and also the auxiliary relations take the form

$$\begin{gathered}
V_Y + (1/4)U + XU_X = 0,\\
XUU_X + VU_Y + c_1(G - (1/2))(H - U^2) = c_1 PU_{YY},\\
XUH_X + VH_Y = c_1 P[c_{01}H_{YY} + c_{10}(U^2)_{YY}],\\
Y = 0: \quad V = U = H - H_B = 0, \quad 0 \leqslant X \leqslant 1,\\
V = U_Y = H_Y = 0, \quad 1 < X,\\
Y = Y_L: \quad U = H = 1, \quad 0 \leqslant X,\\
S_L = \int_0^{Y_L} R_1 dY, \quad S_M = \int_0^{Y_L} \frac{R_1R_2}{R_1 - R_2} dY,\\
R_1 = H - U^2,\ R_2 = H - c_0U^2,\\
D = D_L = S_L/P,\ P = c_2(XD_X + (3/4)D)^2,\\
G = XP_X/P = 2(X^2D_{XX} + (7/4)XD_X)/(XD_X + (3/4)D),\\
c_0 = (\varkappa + 1)/(\varkappa - 1),\ c_1 = (\varkappa - 1)/2\varkappa,\ c_2 = (\varkappa + 1)/2,\\
c_{01} = 1/\mathrm{Pr},\ c_{10} = 1 - c_{01}.
\end{gathered} \tag{2.3}$$

The integral $S_M(X)$ enters as a multiplier in the expression for the Pearson number and determines its sign. Negative values of this integral correspond to the supercritical type of interaction, and positive values correspond to the subcritical type.

The central item in solving this problem is the procedure to seek the thickness distribution of the effective body such that when the procedure is finished the thickness coincides with the viscous layer displacement thickness in the entire computed region, including the boundary layer on the plate and the wake. A detailed description of the procedure for seeking the matching effective body thickness can be found in [7].

3. For the rest of the paper the flow parameters ahead of the shock and behind it are denoted by subscripts – and +, respectively. The first hypothesis made in analyzing the discontinuity is that the characteristic distance in which the sharp pressure variation occurs is small. The second hypothesis, flowing naturally from the first, is that the thickness of the wake behind the discontinuity equals that of the wake ahead of it, to a first approximation. Otherwise the flow scheme would not be self-matching since a variation of wake thickness in a small distance would lead to a large induced pressure and to separation of the plate boundary layer. The third hypothesis is linked to conservation of enthalpy along the stream lines as they pass through the discontinuity. This hypothesis results from using a single system of equations (2.3) in the entire computing region. The appearance of a discontinuous solution is linked to formation of a region with large local gradients. In the gen-

eral case the flow in this region is described by the Euler system of equations. The solution obtained from the original system of equations (2.3) without removing the discontinuity does not satisfy the integral condition of conservation of momentum. In fact, for the action of a finite pressure drop (on a scale characteristic for the boundary layer and the wake) the area of transverse section of an arbitrary stream tube varies by an order of magnitude. In these conditions one must account for the influence of the pressure distribution over the lateral surface of the stream tube. This influence is computed correctly in the system of Euler equations containing a nondegenerate transverse momentum equation. One can postulate different procedures for finding the link between solutions to the left and right of the discontinuity; a description of these procedures falls outside the scope of this paper.

Accounting for the first of the above hypotheses we can reduce the system of equations (2.3) to the form

$$V_Y + (1/4)U + XU_X = 0,$$
$$XUU_X + VU_Y + c_1(G - (1/2))(H - U^2) = 0, \quad XUH_X + VH_Y = 0. \tag{3.1}$$

Converting in system (3.1) from the variables X, Y to the variables X, F, we have

$$XUU_X - \frac{1}{4}FUU_F + c_1\left(G - \frac{1}{2}\right)(H - U^2) = 0, \quad XH_X - \frac{1}{4}FH_F = 0, \quad F = \int_0^Y U\,dY.$$

With the aid of the third hypothesis it is easy to obtain a solution of this system of equations linking the values of the function in front of and behind the discontinuity:

$$H_+ - U_+^2 = P_N^{(\varkappa-1)/\varkappa}(H_- - U_-^2), \quad H_+ = H_-. \tag{3.2}$$

Here the parameter $P_N = P_+/P_-$ describes the pressure drop in the "shock." The first equation of (3.2) can be interpreted as the gas temperature rise in passing through the shock, and the second equation expresses the constancy of total enthalpy along a fixed stream tube.

By considering the as yet unused hypothesis that the wake thickness is constant in transition through the shock we can formulate a relation to determine the so-called critical pressure drop P_N^* leading to change of supercritical interaction to subcritical:

$$S_L(X) = S_N(X, P_N^*), \quad S_L(X) = \int_0^{Y_-} (H_- - U_-^2)\,dY_-,$$
$$S_N(X, P_N) = P_N^{2c_1-1} \int_0^{Y_-} \frac{(H_- - U_-^2)\,U_-}{\left[H_- - P_N^{2c_1}(H_- - U_-^2)\right]^{1/2}}\,dY_-. \tag{3.3}$$

For small values of the pressure drop this change is possible only where the wake flow is transonic on the average and where a change of the thickness of the supersonic part of the wake is accurately compensated for by a change of opposite sign of the thickness of the subsonic part, under the action of even a small pressure perturbation. In the region of developed supercritical flow the change of thickness of the supersonic part under the action of a small pressure perturbation becomes dominant, and now is not compensated for by a change of the thickness of the subsonic part. The action of a finite pressure drop on the supercritical flow leads to a completely different effect. Since the velocity head is less in the subsonic part of the wake than in the supersonic part, the relative variation of its thickness under the action of a finite drop can be compared with the variation of the thickness of the supersonic part. The region of application of the model examined is limited to pressure drop values not large enough to cause reverse flow to appear.

4. Calculations were performed for a monatomic perfect gas with ratio of specific heats $\kappa = c_p{}^0/c_V{}^0 = 5/3$ at unit values of Prandtl number and surface temperature factor. This choice of initial parameters stemmed from the need to approve the method of [7] in the problem formulated, as a typical problem with a discontinuity of the boundary conditions, with a maximum sensitivity of the computed functions to the downstream conditions that is unfavorable from the computing standpoint. The broken lines on the figures show the corresponding

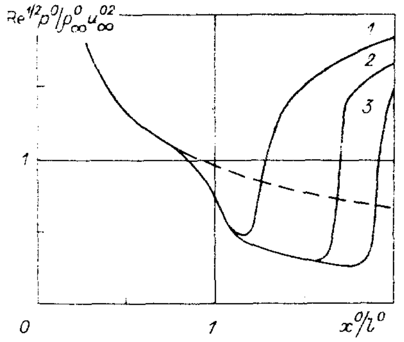

Fig. 1

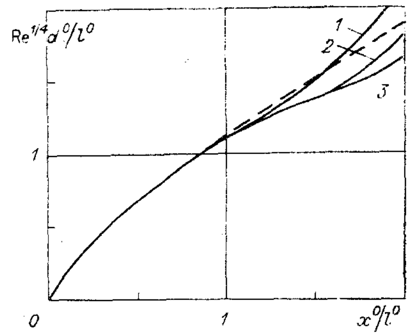

Fig. 2

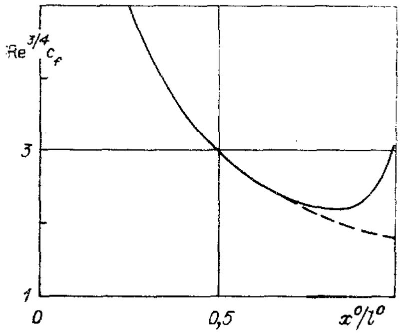

Fig. 3

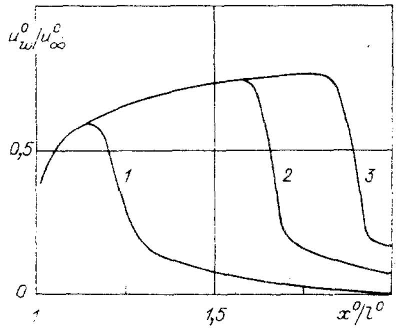

Fig. 4

similarity distributions of the flow functions on a semi-infinite flat plate. The difference mesh had the following parameters: the step sizes and their number in the longitudinal direction were DX = 0.016, L = 125, and in the transverse direction DY = 0.125, M = 50. The difference scheme approximated equations with first order in the longitudinal coordinate and second order in the transverse coordinate.

Figure 1 shows the longitudinal static pressure distribution on the plate and in the wake. The mutual ejector action of the parts of the flow on the two sides leads to a pressure drop in the wake compared with the similarity distribution for a semi-infinite plate. The propagation of perturbations effect [2] leads to the pressure drop associated with the finite length of the plate occurring quite smoothly as one draws near to the trailing edge of the plate, not discontinuously. The extent of this region is approximately four plate lengths. On the other hand, the pressure rise to the previously assigned value at the right hand edge of the computing region (later on we shall conventionally call this the back pressure) occurs discontinuously, i.e., the main portion of the pressure increase occurs in a comparatively short section of the wake. As the back pressure increases this discontinuity moves towards the plate trailing edge, and becomes then increasingly blurred. Blurring of the discontinuity is explained by increased action of viscous forces as one approaches the trailing edge. The back pressure level is described by the parameter P_W. It is the ratio of the previously assigned pressure at the wake section at the right hand edge of the computing region to the pressure that would be formed at the same section but with flow over a semi-infinite plate (in Figs. 1, 2, 4, and 6 lines 1-3 correspond to P_W = 2.75, 2.50, 2.25).

In contrast to the pressure, the boundary layer thickness, which coincides with the displacement thickness in the strong interaction regime, reacts weakly to a change of back pressure (Fig. 2). It increases practically over the entire plate length according to the well-known 0.75 power law [1], and only in the immediate vicinity of the trailing edge does the flow acceleration lead to an insignificant drop in thickness compared with the similarity distribution. An increase of back pressure leads to an increase of wake displacement thickness.

The flow acceleration near the trailing edge leads to a substantial increase of the local friction factor $c_f = \mu^0 \frac{\partial u^0}{\partial y^0} \Big/ \frac{1}{2} \rho^0_\infty u^{02}_\infty$ on the plate surface (Fig. 3). In the range of back

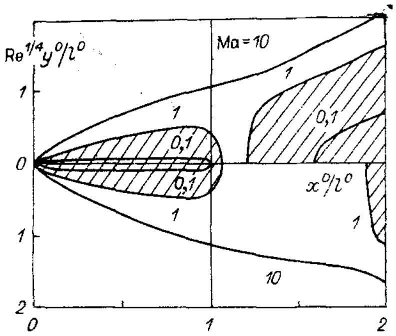

Fig. 5

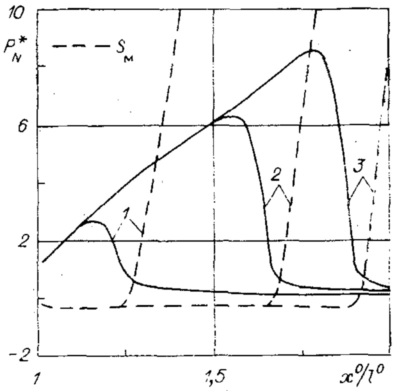

Fig. 6

pressure considered the distribution of surface friction along the plate remains unchanged. The fact is that the flow change from the plate to the wake is accompanied by a strong acceleration, mainly of the near-wall stream. This leads to a blocking off of the flow in a section of the wake located at a small distance from the trailing edge of the plate. Perturbations from the region lying downstream of this section are not propagated upstream. This is why the surface friction on the plate is independent of the back pressure (in the range of change of P_w examined).

An increase of back pressure leads to a discontinuous drop of the longitudinal velocity component on the wake axis $u_w{}^0$ (Fig. 4), and for $P_w = 2.75$ the value of this component is close to zero. Figure 5 shows lines of constant Mach number computed from local values of the sound speed, $Ma = u^0/(\kappa p^0/\rho^0)^{1/2}$. On the lower half surface iso-Mach lines are drawn for $P_w = 2.25$, and on the upper half surface lines are drawn for $P_w = 2.75$. The subsonic regions are shaded. An increase of back pressure leads to transition of a large region of the wake to the subsonic flow regime. Here the shape of the subsonic region adjacent to the plate remains unchanged. The strong acceleration near the trailing edge leads to supersonic flow even in a small distance.

Figure 6 shows the distributions of the critical pressure drop $P_N{}^*$ over sections of the wake, computed for different P_w. With increasing distance from the trailing edge the wake flow is accelerated and its interaction with the outer inviscid flow acquires a more developed supercritical nature. As one would expect, for transition of this flow to the subcritical regime, according to the criterion of Eq. (3.3), with increasing distance into the wake one must add an increasingly unfavorable pressure gradient. We note that the pressure drop sufficient for transcritical transition is unity as in the wake section, where the integral $S_M(X)$ goes to zero, and for which the change of sign corresponds to a change of the type of interaction.

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